

Symmetries of the Poincaré sphere and decoherence matrices

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Abstract

The Stokes parameters form a Minkowskian four-vector under various optical transformations. As a consequence, the resulting two-by-two density matrix constitutes a representation of the Lorentz group. The associated Poincaré sphere is a geometric representation of the Lorentz group. Since the Lorentz group preserves the determinant of the density matrix, it cannot accommodate the decoherence process through the decaying off-diagonal elements of the density matrix, which yields to an increase in the value of the determinant. It is noted that the $O(3,2)$ deSitter group contains two Lorentz subgroups. The change in the determinant in one Lorentz group can be compensated by the other. It is thus possible to describe the decoherence process as a symmetry transformation in the $O(3,2)$ space. It is shown also that these two coupled Lorentz groups can serve as a concrete example of Feynman's rest of the universe.

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1 Introduction

Traditionally the Poincaré sphere plays the central role in the polarization optics [1]. The sphere is also applicable to all two-beams systems with partially coherent phase relations [2, 3]. The sphere has many interesting symmetry properties. Of course, this sphere has three-dimensional rotational symmetries which are well known. What other symmetries does this sphere possesses? This is the question we would like to address in this paper.

Polarization optics can also be formulated in terms of the two-by-two and four-by-four representations of the six-parameter Lorentz group. It was noted that the two-component Jones vector and the four-component Stokes parameters are like the relativistic spinors and the Minkowskian four-vectors, respectively [4, 5]. It is possible to identify the attenuator, rotator, and phase shifter with appropriate transformation matrices of the Lorentz group. This formulation is not restricted to polarization optics. It can be applied to all two-beam systems with coherent or partially coherent phases.

If we use (t, z, x, y) as the Minkowskian four-vector to which four-by-four Lorentz-transformation matrices are applicable, it is possible to write

$$X = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}, \quad (1)$$

with appropriate two-by-two transformation matrices applicable to both sides of this two-by-two representation of the four-vector. These Lorentz transformations are uni-modular transformations, keeping the determinant of the above matrix constant. We can write this in the familiar form

$$t^2 - z^2 - x^2 - y^2 = \text{constant}. \quad (2)$$

If we write the Stokes parameters in this two-by-two form, the matrix becomes the density matrix. This density matrix can also be geometrically represented by the Poincaré sphere. Therefore, the symmetry of the Poincaré sphere is necessarily that of the Lorentz group [3]. In this Lorentzian regime, the determinant of the density matrix is an invariant quantity.

Unlike the Jones vectors, the Stokes parameters, density matrix, and the Poincaré sphere can deal with the lack of coherence between the two beams. The determinant of the density matrix vanishes when the two beams are completely coherent, and it increases as the beams lose coherence. The Lorentzian symmetry of the Poincaré sphere can describe the symmetry with a fixed value of the determinant, but it cannot describe the process in which the determinant changes its value. In other words, we cannot discuss the decoherence process within the framework of the Lorentz group [3].

This decrease in coherence is an irreversible process, and we are tempted to associate this problem with dissipation problems in physics [6]. Of course, the mathematical method closest to group theoretical methods is to introduce the concept of dissipative groups or semi-groups [7]. While this method is quite promising in traditional dissipation problems, we choose to take care of this decoherence problem with a mathematical method which is already familiar to us.

Let us start with a pair of complex numbers a and b . From these numbers, we can construct the density matrix of the form

$$\rho = \begin{pmatrix} aa^* & ab^*e^{-\lambda t} \\ a^*be^{-\lambda t} & bb^* \end{pmatrix}. \quad (3)$$

Indeed, the decay in the off-diagonal elements of this matrix plays a fundamental role in decoherence processes [8, 9].

The determinant of this matrix is

$$aa^*bb^*(1 - e^{-2\lambda t}). \quad (4)$$

This density matrix enjoys the symmetry properties like those of the X matrix given in Eq.(1), since the optical transformations applicable to the Stokes parameters are like Lorentz transformations. However, these determinant-preserving transformations cannot change the t variable.

When $t = 0$, the system is in a pure state, and the determinant is zero. As t increases, the value of the determinant in Eq.(4) increases from zero to aa^*bb^* , and consequently the system becomes decoherent.

The question is whether there is a symmetry group which will accommodate this transition process. We know the Lorentz group cannot, but this does not prevent us from looking for a larger symmetry group. The purpose of the present paper is to show that the deSitter group $O(3, 2)$ accommodates this decoherence process.

This deSitter group is a Lorentz group applicable to a five-dimensional space consisting of three space coordinates and two time coordinates. While the three-dimensional rotation group is applicable to the three space coordinates, the one-parameter two-dimensional rotation group is applicable to the two time coordinates.

Although, this may sound like a mathematical exercise remote from the physical reality, we would like to emphasize that the $O(3, 2)$ deSitter group is already a standard theoretical tool in optical sciences, specifically as a mathematical basis for two-mode squeezed states [10, 11], as well as in the theory of elementary particles together with the $O(4, 1)$ group. As Paul A. M. Dirac noted in 1963, the $O(3, 2)$ group is the fundamental symmetry group for two coupled harmonic oscillators [12]. This two-oscillator system often serves as a mathematical basis for soluble models such as the Lie model in quantum field theory [13] and the Bogoliubov transformations in superconductivity [14].

In this paper we are interested in the fact that the $O(4,1)$ group contains two $O(3,1)$ Lorentz groups, where the two time variables are linearly combined through the one parameter rotation group. We will consider them as two coupled Lorentzian spaces. The loss of coherence in one Lorentzian space will result in the gain in the other space. We shall show that our symmetry model will constitute a concrete example of Feynman's rest of the universe. The first Lorentzian space is the world in which we make physical observations, and the second space belongs to the rest of the universe [15, 16].

In Sec. 2, we review the symmetries of the Stokes parameters and the density matrix. In Sec. 3, we study the symmetries of the Poincaré sphere within the Lorentzian framework and discuss in detail what is possible and what is not possible. In Sec. 4, it is shown that the $O(3,2)$ symmetry can provide a framework for the decoherence process. In Sec 5, we interpret the result of our paper in terms of Feynman's rest of the universe.

2 Stokes Parameters as Minkowskian Four-vectors

Let us start with a plane wave propagating along the z direction. Then, it has polarizations along the x and y directions. We can then write the Jones vector as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A \exp \{i(kz - \omega t)\} \\ B \exp \{i(kz - \omega t)\} \end{pmatrix}. \quad (5)$$

Even though the Jones vector was developed originally for polarized light waves, the formalism can be extended to all two-beam systems such as interferometers [3].

If the two beams are mixed, we use the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad (6)$$

applicable to column vector of Eq.(5).

These two beams can go through two different optical path lengths, resulting in a phase difference. If the phase difference is ϕ , the phase shift matrix is

$$P(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}. \quad (7)$$

When reflected from mirrors, or while going through beam splitters, there are intensity losses for both beams. The rate of loss is not the same for the beams. This results in the attenuation matrix of the form

$$\begin{pmatrix} e^{-\eta_1} & 0 \\ 0 & e^{-\eta_2} \end{pmatrix} = e^{-(\eta_1+\eta_2)/2} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \quad (8)$$

with $\eta = \eta_2 - \eta_1$. This attenuator matrix tells us that the electric fields are attenuated at two different rates. The exponential factor $e^{-(\eta_1 + \eta_2)/2}$ reduces both components at the same rate and does not affect the degree of polarization. The effect of polarization is solely determined by the squeeze matrix

$$S(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}. \quad (9)$$

It was shown in Refs. [4, 3] that repeated applications of the rotation matrices of the form of Eq.(6), shift matrices of the form of Eq.(7) and squeeze matrices of the form of Eq.(9) lead to a two-by-two representation of the six-parameter Lorentz group. The transformation matrix in general takes the form

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (10)$$

applicable to the column vector of Eq.(5), where all four elements are complex numbers with the condition that the determinant of the matrix be one. This matrix contains six free parameters. The above G matrix constitutes the two-by-two representation of the six-parameter Lorentz group, commonly called $SL(2, c)$.

Indeed, the two-component Jones vector provides the representation space for the two-by-two representation of the Lorentz group. However, the Jones vectors cannot describe whether the two-beams are coherent. This is the reason why we have to resort to the coherency matrix

$$C = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \quad (11)$$

with

$$\begin{aligned} S_{11} &= \langle \psi_1^* \psi_1 \rangle, & S_{22} &= \langle \psi_2^* \psi_2 \rangle, \\ S_{12} &= \langle \psi_1^* \psi_2 \rangle, & S_{21} &= \langle \psi_2^* \psi_1 \rangle. \end{aligned} \quad (12)$$

This coherency matrix also serves as the density matrix [15].

Under the influence of the G transformation given in Eq.(10), this density matrix is transformed as

$$\begin{aligned} C' &= G C G^\dagger = \begin{pmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \end{aligned} \quad (13)$$

This leads to the four-by-four transformation

$$\begin{pmatrix} S'_{11} \\ S'_{22} \\ S'_{12} \\ S'_{21} \end{pmatrix} = \begin{pmatrix} \alpha^* \alpha & \gamma^* \beta & \gamma^* \alpha & \alpha^* \beta \\ \beta^* \gamma & \delta^* \delta & \delta^* \gamma & \beta^* \delta \\ \beta^* \alpha & \delta^* \alpha & \beta^* \beta & \delta^* \beta \\ \alpha^* \gamma & \gamma^* \gamma & \alpha^* \delta & \gamma^* \delta \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{22} \\ S_{12} \\ S_{21} \end{pmatrix}. \quad (14)$$

It is sometimes more convenient to use the following combinations of parameters.

$$\begin{aligned} S_0 &= \frac{S_{11} + S_{22}}{\sqrt{2}}, & S_1 &= \frac{S_{11} - S_{22}}{\sqrt{2}}, \\ S_2 &= \frac{S_{12} + S_{21}}{\sqrt{2}}, & S_3 &= \frac{S_{12} - S_{21}}{\sqrt{2}i}. \end{aligned} \quad (15)$$

These four parameters are called the Stokes parameters in the literature [17], usually in connection with polarized light waves. However, as was mentioned before, the Stokes parameters are useful to all two-beam systems. We can write the above expression as

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (S_{11} + S_{22}) \\ (S_{11} - S_{22}) \\ (S_{12} + S_{21}) \\ i(S_{21} - S_{12}) \end{pmatrix}. \quad (16)$$

Then the four-by-four matrix which transforms $(S_{11}, S_{22}, S_{12}, S_{21})$ to (S_0, S_1, S_2, S_3) is

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{22} \\ S_{12} \\ S_{21} \end{pmatrix}. \quad (17)$$

This matrix enables us to construct the transformation matrix applicable to the Stokes parameters, widely known as the Mueller matrix. The transformation matrix applicable to the Stokes parameters of Eq.(15) can be derived from Eq.(14), and its form has been discussed in detail in Refs. [3, 4]. The above Stokes parameters form a Minkowskian four-vector like (t, z, x, y) , and the transformation matrix applicable to the Stokes parameters represents a Lorentz transformation.

The four-by-four representation is like the Lorentz transformation matrix applicable to the space-time Minkowskian vector (t, z, x, y) [3]. This allows us to study space-time symmetries in terms of the Stokes parameters which are applicable to interferometers. Let us first see how the rotation matrix of Eq.(6) is translated into the four-by-four formalism. In this case,

$$\alpha = \delta = \cos(\theta/2), \quad \gamma = -\beta = \sin(\theta/2). \quad (18)$$

Thus, the corresponding four-by-four matrix takes the form

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (19)$$

Let us next see how the phase-shift matrix of Eq.(7) is translated into this four-dimensional space. For this two-by-two matrix,

$$\alpha = e^{-i\phi/2}, \quad \beta = \gamma = 0, \quad \delta = e^{i\phi/2}. \quad (20)$$

For these values, the four-by-four transformation matrix takes the form

$$P(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix}. \quad (21)$$

For the squeeze matrix of Eq.(9),

$$\alpha = e^{\eta/2}, \quad \beta = \gamma = 0, \quad \delta = e^{-\eta/2}. \quad (22)$$

As a consequence, its four-by-four equivalent is

$$S(\eta) = \begin{pmatrix} \cosh \eta & \sinh \eta & 0 & 0 \\ \sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (23)$$

If the above matrices are applied to the four-dimensional Minkowskian space of (t, z, x, y) , the above squeeze matrix will perform a Lorentz boost along the z or S_0 axis with S_0 as the time variable. The rotation matrix of Eq.(19) will perform a rotation around the y or S_3 axis, while the phase shifter of Eq.(21) performs a rotation around the z or the S_1 axis. Matrix multiplications with $R(\theta)$ and $P(\phi)$ lead to the three-parameter group of rotation matrices applicable to the three-dimensional space of (S_1, S_2, S_3) .

The phase shifter $P(\phi)$ of Eq.(21) commutes with the squeeze matrix of Eq.(23), but the rotation matrix $R(\theta)$ does not. This aspect of matrix algebra leads to many interesting mathematical identities which can be tested in laboratories. One of the interesting cases is that we can produce a rotation by performing three squeezes. This aspect is widely known as the Wigner rotation as discussed in the literature.

In this paper, we are interested in studying the time-dependent density matrix of the form

$$C(t) = \begin{pmatrix} S_{11} & S_{12}e^{-\lambda t} \\ S_{21}e^{-\lambda t} & S_{22} \end{pmatrix}. \quad (24)$$

This matrix can be translated into the Minkowskian four-vector

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 e^{-\lambda t} \\ S_3 e^{-\lambda t} \end{pmatrix}. \quad (25)$$

As t increases, the third and fourth component of this Minkowskian four-vector becomes smaller.

Lorentz transformations preserve the $(length)^2$ of the four-vector which in the Minkowskian metric takes the form

$$S_0^2 - S_1^2 - (S_2^2 + S_3^2)e^{-2\lambda t}. \quad (26)$$

This is also the determinant of the density matrix $D(t)$. If this quantity increases as the time t increases, we cannot handle the problem within the framework of the Lorentz group [3].

One option is to assert that this is not a reversible problem and invent a mathematical tool other than group theory [7]. Another approach is to look for a larger group which contains the Lorentz group as a subgroup. This is precisely what we intend to do in this paper. In Sec. 4, we shall introduce the $O(3, 2)$ deSitter group which contains two Lorentz groups. Before getting into the world of the $O(3, 2)$ symmetry, let us study the geometry of the Poincaré sphere in the following section.

3 Lorentz Symmetries of the Poincaré Sphere

The Poincaré sphere has a long history, and its spherical symmetry is well known [1]. The Lorentz group has the three-dimensional rotation group as its subgroup. Thus, the Lorentz symmetry of the Poincaré sphere includes the traditional rotational symmetry. Let us study in this section the symmetries associated with Lorentz boosts.

If we use the expressions of ψ_1 and ψ_2 given in Eq.(5), the density matrix C of Eq.(11) becomes

$$D(t) = \begin{pmatrix} A^2 & AB e^{(-\lambda t - i\phi)} \\ AB e^{(-\lambda t + i\phi)} & B^2 \end{pmatrix}. \quad (27)$$

Here ϕ is the phase difference between $\psi_1^* \psi_2$ and $\psi_1 \psi_2^*$. The λt factor in the exponent describes the loss of coherence. We assume that the off-diagonal terms decrease exponentially in the time variable. The determinant of this density matrix is

$$(AB)^2 (1 - e^{-2\lambda t}). \quad (28)$$

This determinant is zero when $t = 0$, but increases to $(AB)^2$ as t becomes larger.

The corresponding four-vector is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} A^2 + B^2 \\ A^2 - B^2 \\ 2AB(\cos \phi)e^{-\lambda t} \\ 2AB(\sin \phi)e^{-\lambda t} \end{pmatrix}. \quad (29)$$

For a fixed value of t , the geometry of the Poincaré sphere is the geometry defined by the three parameters A, B and ϕ . This sphere consists of two spheres: One is the outer sphere whose radius is the time-like component of the above four-vector

$$s = \frac{(A^2 + B^2)}{2}, \quad (30)$$

and the other is the inner sphere whose radius is the magnitude of the three-vector contained in the four-vector of Eq.(29)

$$r = \frac{1}{2} \sqrt{(A^2 - B^2)^2 + 4(AB)^2 e^{-2\lambda t}}. \quad (31)$$

Then the quantity

$$s^2 - r^2 \quad (32)$$

is Lorentz-invariant, and is equal to the value of the determinant given in Eq.(28). The inner radius is equal to the outer radius when $t = 0$, and becomes $(A^2 - B^2)/2$ as t becomes very large.

We can now introduce a spherical coordinate system with

$$\begin{aligned} r_z &= (A^2 - B^2)/2 = r(\cos \theta), \\ r_x &= AB(\cos \phi)e^{-\lambda t} = r(\sin \theta) \cos \phi, \\ r_y &= AB(\sin \phi)e^{-\lambda t} = r(\sin \theta) \sin \phi. \end{aligned} \quad (33)$$

Then the Lorentz symmetry allows rotations in this three-dimensional system. Now, with the appropriate rotation it is possible to bring four-vector of Eq.(29) to

$$\begin{pmatrix} s \\ r \\ 0 \\ 0 \end{pmatrix}. \quad (34)$$

The rotations do not change the radii of the outer and inner spheres, and r and s remain invariant under the rotations.

However, the Lorentz symmetry allows the Lorentz boosts of the four-vector of Eq.(34) along the $-z$ direction. If we apply the inverse of the boost matrix of Eq.(23), then the four-vector becomes

$$\begin{pmatrix} s(\cosh \eta) - r(\sinh \eta) \\ r(\cosh \eta) - s(\sinh \eta) \\ 0 \\ 0 \end{pmatrix}. \quad (35)$$

This transformation changes the outer and inner radii, but keeps $(s^2 - r^2)$ invariant, as we can see from

$$[s(\cosh \eta) - r(\sinh \eta)]^2 - [r(\cosh \eta) - s(\sinh \eta)]^2 = s^2 - r^2. \quad (36)$$

It is now possible to choose the value of η such that

$$r(\cosh \eta) - s(\sinh \eta) = 0, \quad (37)$$

which leads to $\tanh \eta = r/s$. If this condition is met, the four-vector of Eq.(35) becomes

$$\begin{pmatrix} \sqrt{s^2 - r^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} AB\sqrt{1 - e^{-2\lambda t}} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (38)$$

Indeed, the Lorentz symmetry allows us to bring the Poincaré sphere to a one-number system. We are now tempted to change the value of $(r^2 - s^2)$ in the above expression by changing the time variable t . This is precisely what is not allowed within the framework of the Lorentz group. We shall see whether this can be achieved when symmetry group is enlarged.

4 $O(3,2)$ Symmetry of the Poincaré Sphere

In order to deal with this problem, we introduce the $O(3,2)$ deSitter space with (t, z, x, y, u) where t and u are two time-like variables while allowing two-dimensional rotations in the t and u . As we emphasized in Sec. 1, this group has already been exploited in optical sciences. For instance, it is the fundamental language for two-mode squeezed states [10, 11].

In this deSitter space, we are allowed to have the rotation

$$\begin{pmatrix} \cos \chi & 0 & 0 & 0 & \sin \chi \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\sin \chi & 0 & 0 & 1 & \cos \chi \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m \end{pmatrix} = \begin{pmatrix} m(\sin \chi) \\ 0 \\ 0 \\ 0 \\ m(\cos \chi) \end{pmatrix}. \quad (39)$$

Now, the invariant quantity is

$$t^2 + u^2 - z^2 - x^2 - y^2. \quad (40)$$

As we can see from Eq.(39), if $z = x = y = 0$, this quantity is $(t^2 + u^2) = m^2$, and remains as an invariant in this space. The deSitter space contains two Minkowskian subspaces, namely the spaces of (t, z, x, y) with the invariant of $(t^2 - z^2 - x^2 - y^2)$, and of (u, z, x, y) with the invariant of $(u^2 - z^2 - x^2 - y^2)$.

Let us consider the five-vector $(0, 0, 0, 0, m)$ in this space. The above five-by-five matrix changes this five-vector to

$$(m(\sin \chi), 0, 0, 0, m(\cos \chi)). \quad (41)$$

Thus, in the Minkowskian world of (t, z, x, y) , the invariant quantity is $m^2 \sin^2 \chi$, and $m^2 \cos^2 \chi$ in the Minkowskian space of (u, z, x, y) , where now the four vectors in these spaces are

$$\begin{pmatrix} m(\sin \chi) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} m(\cos \chi) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (42)$$

respectively.

Let us compare the first four-vector of Eq.(42) with the four-vector of Eq.(38). If we identify the parameter $m(\sin \chi)$ in Eq.(42) with $\sqrt{s^2 - r^2}$ of Eq.(38), we have

$$s^2 - r^2 = m^2 \sin^2 \chi. \quad (43)$$

This further allows us to identify m as AB in Eq.(38), and

$$(AB)^2 (\sin \chi)^2 = (AB)^2 (1 - e^{-2\lambda t}), \quad (44)$$

which leads to

$$\cos \chi = e^{-\lambda t}. \quad (45)$$

We concluded in Sec. 3 that the t parameter cannot be changed in the Lorentzian regime. However, we have shown that this decoherence parameter can be identified with the angle variable χ in the deSitter space.

After changing the t variable, we can make inverse transformations to return to the four-vector of the form given in Eq.(29). Indeed, it is gratifying to note that we now have the freedom of changing this time variable with a symmetry operation. In terms of this symmetry parameter, we can write the density matrix as

$$\rho(\chi) = \begin{pmatrix} A^2 & AB e^{-i\phi(\cos \chi)} \\ AB e^{i\phi(\cos \chi)} & B^2 \end{pmatrix}. \quad (46)$$

If $\chi = 0$ and $t = 0$, the system is in a pure state. As t becomes large, the angle χ approaches 90° . Therefore the deSitter parameter χ neatly takes care of the loss of coherence in the two-beam system.

5 Feynman's Rest of the Universe

In this paper, we insinuated two separate Minkowskian spaces by introducing the deSitter space. The first Minkowskian space was defined by the coordinate variables (t, z, x, y) , and the second one by (u, z, x, y) . When we discussed the Lorentzian symmetry of the Poincaré sphere we worked with the first Minkowskian space. How about the second space?

Our analysis would be exactly the same, except that $\sin \chi$ is replaced by $\cos \chi$ as can be seen from Eq.(39). The density matrix in this second space can then be written as

$$\sigma(\chi) = \begin{pmatrix} A^2 & AB e^{-i\phi(\sin \chi)} \\ AB e^{i\phi(\sin \chi)} & B^2 \end{pmatrix}. \quad (47)$$

This density matrix gains coherence as the density matrix of Eq.(46) loses coherence. The determinants of these two density matrices are $(AB)^2 \sin^2 \chi$ and $(AB)^2 \cos^2 \chi$ respectively. The sum of these two determinants is $(AB)^2$ and is independent of angle variable χ . Indeed, these two density matrices or the two Lorentzian subspaces are “coupled” in a Pythagorean manner. What is the meaning of this?

In his book on statistical mechanics [15], Feynman makes the following statement about the density matrix. *When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.*

In order to understand what Feynman said, Han *et al.* used two coupled oscillators to illustrate Feynman's rest of the universe [16]. One of the oscillators is in the world where we make measurements, and the other serves as the rest of the universe. The two coupled oscillators form the entire universe.

By working with two separate Lorentz subgroups of the deSitter group, we divided the universe into the world where we measure the degree of decoherence and the hidden world which is still controlling the events in its counterpart. The $O(3, 2)$ deSitter world constitutes the entire universe.

It is gratifying to note that the present paper provides another illustrative example of Feynman's rest of the universe.

Concluding Remarks

It has been widely believed that the decoherence problem could not be treated as a symmetry problem. In this paper, we have presented a different view, using an extra time-like dimension in the Lorentz group. The deSitter group we used has been one of the standard tools in relativistic quantum mechanics [18] and elementary particle

physics including one of the most recent models in string theory [19]. Also, this group is not new in optical sciences. In 1963, Paul A. M. Dirac observed that the deSitter group $O(3, 2)$ serves as a symmetry group for coupled harmonic oscillators [12]. This group is the fundamental scientific language for two-mode squeezed states of light [10, 11]. We are thus not carrying the burden of introducing a new mathematical device in this paper.

Of course, a more challenging problem is to compute the decay parameter λ from dynamical considerations, but this is beyond the scope of the present paper dealing solely with symmetry problems. However, this symmetry property may be helpful in formulating dynamical problems in the future.

As we noted in Sec. 5, the $O(3, 2)$ group can serve as an illustrative example of Feynman's rest of the universe. One Lorentz subgroup represents the system under examination, while the other appears as the rest of the universe. As Feynman noted, it is more satisfying to understand the entire system including the rest of the universe.

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